J. Fluid Mech. (1989), *vol.* 208, *pp.* 115–126
Printed in Great Britain

Lagrangian approach to the mean-field electrodynamics for turbulent fluids with arbitrary conductivities

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(Received 20 June 1988 and in revised form 2 February 1989)

A modification is made to the traditional Lagrangian approach to the derivation of the mean EMF of turbulent fluids which allows for finite conductivities. Consideration is confined to the case of homogeneous, isotropic but generally mirror-non-invariant and compressible turbulence. The eddy magnetic diffusivity and the coefficient α of the alpha-effect are expressed in terms of statistical moments of displacements of adjacent particles which undergo convective transport and microscopic diffusion in a turbulent flow. These expressions, being valid for arbitrary conductivities, reproduce known results in the cases of both very large and very small magnetic Reynolds numbers. Difficulties and advantages of the use of the results obtained for evaluations of the mean EMF are discussed.

1. Introduction

Mean-field electrodynamics considers the evolution of the mean magnetic field $\boldsymbol{B}$ in turbulent flows of conducting fluids. Formally, the problem is to find an adequate expression for the mean EMF,

$$\boldsymbol{\varepsilon} = \overline{\boldsymbol{u} \times \boldsymbol{H}} \tag{1.1}$$

that contributes the equation for the field $\boldsymbol{B}$:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \overline{(\boldsymbol{u} \times \boldsymbol{H})} + \eta \nabla^2 \boldsymbol{B}, \tag{1.2}$$

where $\boldsymbol{u}$ is hydrodynamic velocity; $\boldsymbol{H}$ is the 'full' field, i.e. the superposition of the mean, $\boldsymbol{B}$, and fluctuating, $\boldsymbol{h}$ ($\bar{\boldsymbol{h}} = 0$), magnetic fields: $\boldsymbol{H} = \boldsymbol{B} + \boldsymbol{h}$; η is the microscopic magnetic diffusivity; the bar means averaging over an ensemble of realizations of random flow.

The mean EMF (1.1) can be conveniently expanded in powers of spatial derivatives of the mean magnetic field:

$$\boldsymbol{\varepsilon} = \alpha \boldsymbol{B} - \eta_{\mathrm{T}} \nabla \times \boldsymbol{B} + \ldots, \tag{1.3}$$

where dots signify spatial derivatives of the field $\boldsymbol{B}$ of second and higher orders which are usually neglected. We assume for simplicity in (1.3) and in what follows that the turbulence is statistically homogeneous and isotropic and the mean velocity equals zero. It follows from (1.3) that the coefficients of eddy diffusion, η_{T}, and generation, α, of the magnetic field are sufficient to determine the mean EMF for this case.

The literature on mean-field electrodynamics, which is vast and summarized in a range of monographs (e.g. Moffatt 1978; Krause & Rädler 1981; Zeldovich, Ruzmaikin & Sokoloff 1983), contains rigorous derivations of the mean EMF in the quasi-linear approximation (valid for the cases of a delta-correlated-in-time random flow and of small magnetic Reynolds numbers) and for the case of infinite conductivity. For the latter case exact relations were found which express the coefficients η_T and α in terms of fluid particle displacements in a turbulent flow (Moffatt 1974). The principal goal of present paper is to generalize these results to allow for finite conductivities. With this aim in mind, we modify the traditional Lagrangian approach and express the coefficients of expansion (1.3) in terms of displacements of two adjacent particles which experience not only convective transport but a special kind of microscopic diffusion in a turbulent flow.

These expressions are valid for arbitrary magnetic Reynolds numbers. They are very similar in structure to the results by Moffatt (1974). However, the meaning of the averaging procedure used in them changes when magnetic diffusion is taken into account. We shall see that the expressions found for α, (3.13), and η_T, (3.17), reduce to well-known results in the cases of both very large and very small magnetic Reynolds numbers.

In §2 we shall briefly consider the case of perfect conductivity. The joint distribution function of two adjacent fluid particles is introduced and is shown to contain enough information to determine the behaviour of the magnetic field. Such a reformulation of magnetic field dynamics in terms of the distribution function of fluid particles is well suited to the subsequent allowance for finite conductivity. In §3 we modify the equation for this distribution function by adding a diffusional term and show that this modification is strictly equivalent to allowance for finite molecular diffusion of the magnetic field. The eddy magnetic diffusivity and the coefficient α of the alpha-effect are then expressed in terms of statistical moments of spatial displacements of two adjacent particles which experience not only convective transport but also a special kind of microscopic diffusion. In §4 the quasi-linear equation for the joint distribution function of two neighbouring particles is derived and then solved to demonstrate that the Lagrangian expressions for η_T and α of §3 reproduce known results for the case of low conductivity. The results obtained are discussed in §4.

2. Lagrangian approach and distribution functions of fluid particles for perfectly conducting fluids

This Section briefly considers the application of the joint distribution functions of two neighbouring fluid particles to a description of the dynamics of magnetic fields frozen to perfectly conducting turbulent fluids. Some additional information may be found in a recent paper by Vainshtein & Kichatinov (1986). The derivations of the present Section constitute a basis for the allowance for finite conductivity made in §3.

The induction equation for perfectly conducting fluids,

$$\frac{\partial \boldsymbol{H}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{H}), \tag{2.1}$$

is known to have the exact solution in Lagrangian variables:

$$H_\mu(\boldsymbol{X}(\boldsymbol{a},t),t) = \frac{\rho(\boldsymbol{X}(\boldsymbol{a},t),t)}{\rho(\boldsymbol{a},0)} \frac{\partial X_\mu(\boldsymbol{a},t)}{\partial a_\alpha} H_\alpha(\boldsymbol{a},0), \tag{2.2}$$

where $\boldsymbol{X}(\boldsymbol{a},t)$ is the position at time t of a fluid particle, which started at $\boldsymbol{a}$ at $t = 0$, and ρ is fluid density. As in the previous paper (Vainshtein & Kichatinov 1986), we change from Lagrangian to Eulerian variables $\boldsymbol{x}$ and t to exclude density from the Lagrangian solution (2.2) (a is now a function of variables $\boldsymbol{x}$ and t) and introduce the vector potential $\mathscr{A}$ for the field $\boldsymbol{H}$ which somewhat simplifies the solution (2.2):

$$\mathscr{A}_\mu(\boldsymbol{x}, t) = \frac{\partial a_\alpha(\boldsymbol{x}, t)}{\partial x_\mu} \mathscr{A}_\alpha(\boldsymbol{a}(\boldsymbol{x}, t), 0), \qquad (2.3)$$

$$\boldsymbol{H} = \boldsymbol{\nabla} \times \mathscr{A}.$$

Consider now the microscopic (not averaged over an ensemble of random flows) probability density of initial positions, $\boldsymbol{y}$ and $\boldsymbol{y}'$, of two liquid particles:

$$f(\boldsymbol{x}, \boldsymbol{x}' \,|\, \boldsymbol{y}, \boldsymbol{y}', t) = \delta(\boldsymbol{y} - \boldsymbol{a}(\boldsymbol{x}, t))\, \delta(\boldsymbol{y}' - \boldsymbol{a}(\boldsymbol{x}', t)).$$

For each realization of the flow, f is the probability density of initial (at $t = 0$) positions, $\boldsymbol{y}$ and $\boldsymbol{y}'$, of two particles provided that at the present moment t they are at $\boldsymbol{x}$ and $\boldsymbol{x}'$ respectively. The function f is shown in Appendix A to satisfy the equation

$$\frac{\partial f}{\partial t} + \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{\nabla} f + \boldsymbol{u}(\boldsymbol{x}', t) \cdot \boldsymbol{\nabla}' f = 0 \qquad (2.4)$$

with the initial condition

$$f_{t=0} = \delta(\boldsymbol{x} - \boldsymbol{y})\, \delta(\boldsymbol{x}' - \boldsymbol{y}'), \qquad (2.5)$$

where $\boldsymbol{\nabla}' = \partial/\partial \boldsymbol{x}'$. On using the function f, the solution (2.3) can be written as

$$\mathscr{A}_\mu(\boldsymbol{x}, t) = \lim_{\boldsymbol{x}' \to \boldsymbol{x}} \nabla'_\mu \int f(\boldsymbol{x}, \boldsymbol{x}' \,|\, \boldsymbol{y}, \boldsymbol{y}', t)\, y'_\alpha \mathscr{A}_\alpha(\boldsymbol{y}, 0)\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{y}'. \qquad (2.6)$$

It is easy to show that (2.6) and (2.4) are equivalent to the initial induction equation (2.1).

Indeed, successive time-differentiation of (2.6), use of (2.4) to exclude $\partial f/\partial t$, and application of the relation

$$\lim_{\boldsymbol{x}' \to \boldsymbol{x}} (\boldsymbol{\nabla} + \boldsymbol{\nabla}') f = \boldsymbol{\nabla} \lim_{\boldsymbol{x}' \to \boldsymbol{x}} f \qquad (2.7)$$

yield

$$\frac{\partial \mathscr{A}_\mu}{\partial t} + u_\alpha \nabla_\alpha \mathscr{A}_\mu + \mathscr{A}_\alpha \nabla_\mu u_\alpha = 0.$$

Taking curl of this equation returns us to the initial induction equation (2.1).

Hence, we may conclude that the distribution function of two fluid particles, defined by (2.4) and (2.5), contains enough information on the fluid motion to define the magnetic field dynamics in the case of perfect conductivity. Actually, it is sufficient to consider particles very close together because the limit $\boldsymbol{x}' \to \boldsymbol{x}$ is taken in (2.6). Vector properties of the magnetic field are accounted for by the relative positions of the two particles.

The relation (2.6) may be used to express the coefficients α and η_{T} of (1.3) in terms of statistical moments of displacements of the fluid particles. However, we postpone these derivations to §3 where finite conductivity will be taken into account.

3. Lagrangian approach for finite conductivity

Let us now allow for microscopic diffusion of the magnetic field. We shall try to reach this goal within the Lagrangian approach of the proceding Section and keep (2.6) unchanged. It may be hoped that finite conductivity will be taken into account by inclusion of finite diffusion of the two liquid particles considered in §2. The new distribution function $\mathscr{P}(\boldsymbol{x},\boldsymbol{x}'\,|\,\boldsymbol{y},\boldsymbol{y}',t)$ of these not only convected but also diffusing particles satisfies a modified equation (2.4). This modification consists in introducing a diffusional term, $\eta(\boldsymbol{\nabla}+\boldsymbol{\nabla}')^2\mathscr{P}$, into the right-hand side of (2.4) (the initial condition (2.5) remains unchanged):

$$\frac{\partial\mathscr{P}}{\partial t}+\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}\mathscr{P}+\boldsymbol{u}(\boldsymbol{x}',t)\cdot\boldsymbol{\nabla}'\mathscr{P}=\eta(\boldsymbol{\nabla}+\boldsymbol{\nabla}')^2\mathscr{P}, \tag{3.1}$$

$$\mathscr{P}_{t=0}=\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{x}'-\boldsymbol{y}').$$

Equation (2.6) now reads

$$\mathscr{A}_\mu(\boldsymbol{x},t)=\lim_{x'\to x}\nabla'_\mu\int\mathscr{P}(\boldsymbol{x},\boldsymbol{x}'\,|\,\boldsymbol{y},\boldsymbol{y}',t)\,y'_\alpha\,\mathscr{A}_\alpha(\boldsymbol{y},0)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{y}'. \tag{3.2}$$

Successive time-differentiation of (3.2) and using (3.1) yields

$$\frac{\partial\mathscr{A}_\mu}{\partial t}=\lim_{x'\to x}\nabla'_\mu\int[-\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}\mathscr{P}-\boldsymbol{u}(\boldsymbol{x}',t)\cdot\boldsymbol{\nabla}'\mathscr{P}+\eta(\boldsymbol{\nabla}+\boldsymbol{\nabla}')^2\mathscr{P}]\,y'_\alpha\,\mathscr{A}_\alpha(\boldsymbol{y},0)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{y}'. \tag{3.3}$$

On using (2.7), after some algebra, we obtain from (3.3) the following equation:

$$\frac{\partial\mathscr{A}_\mu}{\partial t}=-u_\alpha\nabla_\alpha\mathscr{A}_\mu-\mathscr{A}_\alpha\nabla_\mu u_\alpha+\eta\nabla^2\mathscr{A}_\mu.$$

Taking curl of this equation gives

$$\frac{\partial\boldsymbol{H}}{\partial t}=\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{H})+\eta\nabla^2\boldsymbol{H}. \tag{3.4}$$

Therefore, relation (3.2) with the function $\mathscr{P}$ obeying (3.1) is strictly equivalent to the induction equation (3.4) with arbitrary magnetic diffusivity.

Let us consider (3.1) in more detail. The physical content of this equation becomes more transparent when new variables, $\boldsymbol{r}=\frac{1}{2}(\boldsymbol{x}+\boldsymbol{x}')$ and $\Delta\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{x}'$, are used:

$$\frac{\partial\mathscr{P}}{\partial t}+u_\mu(\boldsymbol{r},t)\frac{\partial\mathscr{P}}{\partial r_\mu}+\Delta r_\alpha\frac{\partial u_\mu(\boldsymbol{r},t)}{\partial r_\alpha}\frac{\partial\mathscr{P}}{\partial\,\Delta r_\mu}=\eta\frac{\partial^2\mathscr{P}}{\partial r_\mu\,\partial r_\mu}. \tag{3.5}$$

Only terms of first order in Δr are retained in this equation because the limit $\Delta r\to 0$ is taken in (3.2). Equation (3.5) describes two adjacent particles with their 'centre of gravity', $\boldsymbol{r}$, undergoing convective transport with local velocity $\boldsymbol{u}(\boldsymbol{r},t)$ and diffusion with coefficient η. The changes of relative position, $\Delta\boldsymbol{r}$, of these particles are brought about exclusively by velocity variations, $\Delta\boldsymbol{u}=(\partial\boldsymbol{u}/\partial r_\mu)\,\Delta r_\mu$, on the scale Δr, which can be considered to be the smallest one between spatial scales typical of the problem. It should be noted that there is no microscopic diffusion in the relative position $\Delta\boldsymbol{r}$ of the particles in (3.5). Thus, adjacent particles remains close to each other at all times; turbulent separation of particles (Batchelor 1950) is an effect of second order in Δr and plays no role in the problem considered.

Ensemble averaging of (3.2) yields

$$A_\mu(\boldsymbol{x},t)=\lim_{x'\to x}\nabla'_\mu\int P(\boldsymbol{x},\boldsymbol{x}'\,|\,\boldsymbol{y},\boldsymbol{y}',t)\,y'_\alpha A_\alpha(\boldsymbol{y},0)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{y}', \tag{3.6}$$

where $\boldsymbol{A}=\overline{\mathscr{A}}$ and $P=\overline{\mathscr{P}}$ are respectively the mean vector potential and the macroscopic distribution function of the initial positions of two particles. Equation

(3.6) relates initial and subsequent mean fields in terms of Lagrangian characteristics of turbulent flow. Note that (3.5) for the microscopic distribution function $\mathscr{P}$ takes finite conductivity into account. In common with the case of $\eta = 0$, it suffices to know the statistics of displacements of two adjacent particles to describe the dynamics of the mean magnetic field. The only complication involved by allowing for finite magnetic viscosity, $\eta \neq 0$, is the inclusion of the particles' centre-of-gravity diffusion with diffusivity equal to η.

Finite conductivity has been modelled previously in the framework of the Lagrangian approach by adding the delta-correlated-in-time spatially independent velocity field to the turbulent flow (Molchanov, Ruzmaikin & Sokoloff 1984; Drummond & Horgan 1986). Such an approach, though having no strict foundation, seems plausible on physical grounds. The delta-correlated-in-time spatially independent random flow seems to represent adequately the centre-of-gravity diffusion of fluid particles without a diffusion in relative positions of these particles.

The derivation of the equation for the mean magnetic field may be approached by averaging (3.1) or (3.5) to derive an equation for the distribution function P and by subsequent use of (3.6). However, averaging of the equation for $\mathscr{P}$ does not seem to be an easier task than averaging of the original induction equation (3.4). At the same time, nothing prevents us from expressing coefficients of the expansion (1.3) directly in terms of the distribution function P, the physical meaning of which is quite clear. This can be done by using (3.6). Actually, such an approach has been used by Moffatt (1974) to treat the case of infinite conductivity.

A method of trial field distributions similar to that applied by Kraichnan (1976) will be followed below. Initial field distributions will be adopted, which are simple enough to enable the solution of (1.2), for arbitrary coefficients α and η_{T} in (1.3) to be found. Comparison of these solutions with (3.6) yields expressions for α and η_{T} in terms of statistical moments of the function P.

Some introductory remarks are needed before starting these derivations. The notation $\boldsymbol{\xi}(\boldsymbol{x},t) = \boldsymbol{x}-\boldsymbol{a}(\boldsymbol{x},t)$ will be used for the particle displacements during time t as a function of position $\boldsymbol{x}$. Angle brackets, $\langle\ \rangle$, means averaging over the distribution P. For example,

$$\left\langle \xi_\mu \frac{\partial}{\partial x_\alpha} \xi_\nu \right\rangle = \lim_{x' \to x} \nabla'_\alpha \int (x-y)_\mu (x'-y')_\nu P(\boldsymbol{x},\boldsymbol{x}' \,|\, \boldsymbol{y},\boldsymbol{y}',t)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{y}'.$$

Note that the procedures for averaging over the distribution P and over an ensemble of random flows (denoted above by an overbar) do not coincide. They do so, however, when there is no microscopic diffusion ($\eta = 0$). Homogeneity of the turbulence considered implies in particular that

$$\langle g(\boldsymbol{\xi})\, \nabla\varphi(\boldsymbol{\xi})\rangle = -\langle \varphi(\boldsymbol{\xi})\, \nabla g(\boldsymbol{\xi})\rangle, \tag{3.7}$$

where g and φ are arbitrary functions (or tensors of arbitrary rank) of the displacements. Indeed,

$$\begin{aligned}\langle g(\boldsymbol{\xi})\, \nabla\varphi(\boldsymbol{\xi})\rangle &= \lim_{x' \to x} \nabla' \int g(\boldsymbol{x}-\boldsymbol{y})\, \varphi(\boldsymbol{x}'-\boldsymbol{y}')\, P\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{y}' \\ &= \lim_{x' \to x} [(\nabla + \nabla') - \nabla] \int g(\boldsymbol{x}-\boldsymbol{y})\, \varphi(\boldsymbol{x}'-\boldsymbol{y}')\, P\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{y}' \\ &= \nabla \langle g(\boldsymbol{\xi})\varphi(\boldsymbol{\xi})\rangle - \langle \varphi(\boldsymbol{\xi})\, \nabla g(\boldsymbol{\xi})\rangle,\end{aligned}$$

and (3.7) follows because $\nabla\langle \varphi g\rangle$ equals zero for homogeneous turbulence. Equation (3.7) and isotropy of the turbulence lead to following relations:

$$\langle \xi_\mu \nabla_\alpha \xi_\nu\rangle = \tfrac{1}{6}\langle \xi\cdot \operatorname{curl}\xi\rangle\,\epsilon_{\mu\alpha\nu},$$
$$\langle \xi_\mu \xi_\alpha \nabla_\nu \xi_\sigma\rangle = \tfrac{1}{12}\langle \xi^2 \operatorname{div}\xi\rangle\,(2\delta_{\mu\alpha}\delta_{\nu\sigma}-\delta_{\mu\nu}\delta_{\alpha\sigma}-\delta_{\mu\sigma}\delta_{\alpha\nu}). \tag{3.8}$$

It is convenient methodologically to use an expression for the mean magnetic field instead of (3.6) for the mean vector potential. On taking curl of (3.6), we find

$$B_\mu(\boldsymbol{x},t) = \epsilon_{\mu\alpha\beta}\lim_{x'\to x}\nabla_\alpha\nabla'_\beta\int P(\boldsymbol{x},\boldsymbol{x}'\,|\,\boldsymbol{y},\boldsymbol{y}',t)\,y'_\sigma A_\sigma(\boldsymbol{y},0)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{y}'. \tag{3.9}$$

Finally, note that the field distributions considered below are algebraic functions of coordinates of not higher than second order. Thus, the terms with spatial derivatives of second and higher orders, which are shown by dots in (1.3) do not have any role.

Let the field distribution at the initial moment $t = 0$ be

$$\boldsymbol{B}(\boldsymbol{r},0) = \tfrac{1}{2}\boldsymbol{\omega}\times\boldsymbol{r}, \tag{3.10}$$

where $\boldsymbol{\omega}$ is a constant vector of suitable dimension. On solving the induction equation for the mean field, we find

$$\boldsymbol{B}(\boldsymbol{r},t) = \tfrac{1}{2}\boldsymbol{\omega}\times\boldsymbol{r}+\boldsymbol{\omega}\int_0^t \alpha(\tau)\,\mathrm{d}\tau. \tag{3.11}$$

The vector potential for the field (3.10) is

$$\boldsymbol{A}(\boldsymbol{r},0) = -\tfrac{1}{4}\boldsymbol{\omega}r^2. \tag{3.12}$$

Substitution of (3.11) and (3.12) into (3.9) forming a scalar product with $\boldsymbol{\omega}$ yield

$$\int_0^t \alpha(\tau)\,\mathrm{d}\tau = -\tfrac{1}{4}e_\mu e_\alpha \epsilon_{\mu\beta\gamma}\lim_{x'\to x}\nabla_\beta\nabla'_\gamma\int P(\boldsymbol{x},\boldsymbol{x}'\,|\,\boldsymbol{y},\boldsymbol{y}',t)\,y'_\alpha y^2\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{y}',$$

where $\boldsymbol{e} = \boldsymbol{\omega}/\omega$ is a unit vector. Let us differentiate this expression with respect to time and represent $\boldsymbol{y}$ and $\boldsymbol{y}'$ in the integrand as $\boldsymbol{y} = \boldsymbol{x}-\boldsymbol{\xi}$. After some algebra, with the use of (3.7) and (3.8), we find

$$\alpha(t) = -\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi\cdot\operatorname{curl}\xi\rangle. \tag{3.13}$$

Now let the initial field distribution be

$$\boldsymbol{B}(\boldsymbol{r},0) = \boldsymbol{\omega}(\delta_{\mu\alpha}-e_\mu e_\alpha)\,r_\mu r_\alpha. \tag{3.14}$$

At later moments we have

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{r},t) &= \boldsymbol{\omega}(\delta_{\mu\alpha}-e_\mu e_\alpha)\,r_\mu r_\alpha \\ &\quad +4\boldsymbol{\omega}\int_0^t(\eta_{\mathrm{T}}(\tau)+\eta)\,\mathrm{d}\tau+2(\boldsymbol{r}\times\boldsymbol{\omega})\int_0^t\alpha(\tau)\,\mathrm{d}\tau \\ &\quad -4\boldsymbol{\omega}\int_0^t\mathrm{d}\tau\int_0^\tau \mathrm{d}\tau'\alpha(\tau)\,\alpha(\tau').\end{aligned} \tag{3.15}$$

The vector potential for the initial field (3.14) is

$$\boldsymbol{A}(\boldsymbol{r},0) = \tfrac{1}{4}(\boldsymbol{\omega}\times\boldsymbol{r})\,(\delta_{\mu\alpha}-e_\mu e_\alpha)\,r_\mu r_\alpha. \tag{3.16}$$

Let us substitute (3.15) and (3.16) into (3.9), form the scalar product of the result with the vector $\boldsymbol{\omega}$, and differentiate the resulting expression with respect to time. After some algebra, with the use of (3.7) and (3.8), we find

$$\eta_{\mathrm{T}}(t)+\eta = \frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\langle\xi^2\rangle - \frac{1}{12}\frac{\mathrm{d}}{\mathrm{d}t}\langle\xi^2 \operatorname{div}\boldsymbol{\xi}\rangle + \alpha(t)\int_0^t \alpha(\tau)\,\mathrm{d}\tau. \tag{3.17}$$

where α is defined by (3.13).

Expressions (3.13) and (3.17) define the coefficients of the expansion (1.3) for the mean EMF and thus close the induction equation (1.2).

A more general but less transparent approach to the derivation of the mean EMF is demonstrated in Appendix B. The expansion (1.3) is not used there and, in principle, fields may be considered of spatial scale not large compared with the typical scale of the turbulence. For example, the coefficient γ in the third term, $\gamma\nabla^2\boldsymbol{B}$, of (1.2) may be found from the results of Appendix B to be

$$\begin{aligned}\gamma = \frac{1}{72}\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{\xi}\cdot\operatorname{curl}\boldsymbol{\xi}\rangle\,(\tfrac{1}{9}\langle\boldsymbol{\xi}\cdot\operatorname{curl}\boldsymbol{\xi}\rangle^2 \\ -\langle\xi^2\operatorname{div}\boldsymbol{\xi}\rangle) - \frac{1}{12}\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{5}\langle\xi^2\boldsymbol{\xi}\cdot\operatorname{curl}\boldsymbol{\xi}\rangle \\ -\tfrac{1}{3}\langle\xi^2\rangle\langle\boldsymbol{\xi}\cdot\operatorname{curl}\boldsymbol{\xi}\rangle).\end{aligned} \tag{3.18}$$

In the limit of perfect conductivity, (3.13) and (3.17) are the same as the results of Moffatt (1974). The second term in the right-hand side of (3.17) differs in sign from that given by Moffatt (1978). However, the discrepancy is illusory. The point here is that Moffatt considered an incompressible fluid and averaged over final positions of fluid particles. The distribution function of final positions of two particles obeys the same equation (i.e. (2.4)) as the function f does when the velocity field is divergence-free. This is shown in Appendix A. Hence, the corresponding macroscopic distributions should coincide in this case and be invariant with respect to the transformation $\boldsymbol{x}, \boldsymbol{x}' \leftrightarrow \boldsymbol{a}, \boldsymbol{a}'$. It is necessary to do this transformation in (3.13) and (3.17) to reproduce Moffatt's results. This transforms $\partial/\partial\boldsymbol{x}$ into $\partial/\partial\boldsymbol{a}$ and changes the sign of the displacements $\boldsymbol{\xi}$, thus reversing the sign of the second term on the right-hand side of (3.17).

4. The quasi-linear approximation

Let us now show that (3.13) and (3.17) can serve to reproduce the results of quasi-linear theory. Consider first the equation for the function P. The quasi-linear approximation is widely used and well known. (It is sometimes named the 'first-order smoothing approximation' or the 'second-order correlation approximation'.) Therefore, we sidestep the derivation of the quasi-linear equation for P. It is sufficient to note that this equation results from ensemble averaging of (3.5) and substitution of the fluctuating distribution function, $\delta\mathscr{P}$, obtained from the linearized equation (3.5), into the second-order moment $\overline{\boldsymbol{u}\delta\mathscr{P}}$. The resulting equation is

$$\frac{\partial P}{\partial t} = (\eta+b)\frac{\partial^2 P}{\partial r_\mu \partial r_\mu} + a, \epsilon_{\mu\alpha\nu}\Delta r_\mu \frac{\partial^2 P}{\partial r_\alpha \partial\,\Delta r_\nu}. \tag{4.1}$$

The coefficients a and b are expressed in terms of spectra of intensity, $E(k,\omega)$, and helicity, $H(k,\omega)$, of fluctuating velocities:

$$\left.\begin{aligned} b &= \frac{\eta}{3}\iint_0^\infty \frac{E(k,\omega)\,k^2}{\eta^2k^4+\omega^2}\,\mathrm{d}k\,\mathrm{d}\omega, \\ a &= -\frac{\eta}{3}\iint_0^\infty \frac{H(k,\omega)\,k^2}{\eta^2k^4+\omega^2}\,\mathrm{d}k\,\mathrm{d}\omega, \\ \overline{u^2} &= \iint_0^\infty E(k,\omega)\,\mathrm{d}k\,\mathrm{d}\omega, \quad \overline{\boldsymbol{u}\cdot\operatorname{curl}\boldsymbol{u}} = \int_0^\infty H(k,\omega)\,\mathrm{d}k\,\mathrm{d}\omega. \end{aligned}\right\} \tag{4.2}$$

The solution of (4.1) can be expressed in the form of the integral

$$\begin{aligned} P(\boldsymbol{r},\Delta\boldsymbol{r}\,|\,\boldsymbol{r}_0,\Delta\boldsymbol{r}_0,t) = \int \exp\{&-(\eta+b)\,k^2t \\ &+\mathrm{i}z_\mu\,\Delta r_\alpha[\delta_{\mu\alpha}\cosh(akt)+k_\mu k_\alpha(1-\cosh(akt))/k^2] \\ &+\epsilon_{\mu\alpha\nu}z_\mu k_\alpha\,\Delta r_\nu\sinh(akt)/k-\mathrm{i}\boldsymbol{z}\cdot\Delta\boldsymbol{r}_0 \\ &+\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)\}\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{z}/(2\pi)^6. \end{aligned} \tag{4.3}$$

Integrations over wave vectors in (4.3) result from the use of Fourier transformations when solving (4.1).

On using the distribution function (4.3), we find

$$\langle\xi^2\rangle = 6(b+\eta)\,t, \quad \langle\boldsymbol{\xi}\cdot\operatorname{curl}\boldsymbol{\xi}\rangle = -6at,$$
$$\langle\xi^2\operatorname{div}\boldsymbol{\xi}\rangle = 6a^2t^2.$$

Substitution of these expressions into (3.13) and (3.17) leads to the well-known (cf. Krause & Rädler 1981) results of quasi-linear theory:

$$\left.\begin{aligned} \eta_{\mathrm{T}} = b &= \frac{\eta}{3}\iint_0^\infty \frac{E(k,\omega)\,k^2}{\eta^2k^4+\omega^2}\,\mathrm{d}k\,\mathrm{d}\omega, \\ \alpha = a &= -\frac{\eta}{3}\iint_0^\infty \frac{H(k,\omega)\,k^2}{\eta^2k^4+\omega^2}\,\mathrm{d}k\,\mathrm{d}\omega. \end{aligned}\right\} \tag{4.4}$$

Certainly, these results are obtainable in an easier way; e.g. by time-differentiation of (3.6) and subsequent use of (4.1). However, the derivations done above clearly demonstrate in what way different non-stationary terms in (3.17) compensate for each other to yield the time-independent result.

Hence, it has been shown that Lagrangian expressions (3.13) and (3.17) lead to the quasi-linear result (4.4) and are particularly valid in the case of low conductivity.

5. Discussion

Above, we have expressed the coefficients α and η_{T} in terms of displacements of two adjacent particles in a turbulent flow. These relations are valid for arbitrary magnetic Reynolds numbers and link different theories developed for the opposite limits of infinite and low conductivities. In the case of perfect conductivity the results obtained coincide with those reported by Moffatt (1974). At the same time, our findings have been shown to reproduce quasi-linear expressions for α and η_{T} which are valid, in particular, in the case of small magnetic Reynolds numbers.

Derivations of the explicit expressions for the distribution function P, over which the liquid particle displacements are averaged when quantities α and η_T are evaluated, is a very complicated problem. At the same time, the physical processes which govern the dynamics of the distribution function P are known and relatively simple. The centre of gravity $\boldsymbol{r}$ of neighbouring particles experiences convective transport with local velocity $\boldsymbol{u}(\boldsymbol{r}, t)$ and diffusion, the diffusivity η of which coincides with the coefficient of microscopic diffusion of the magnetic field. The relative position $\Delta\boldsymbol{r}$ of the particles changes owing to fluid velocity variations on the scale Δr. From this picture it becomes clear that microscopic diffusion is efficient when the time $\tau_d = l^2/\eta$ (l is the correlation length of fluctuations) equals in order of magnitude, or is smaller than, the smaller of the correlation time τ_c of velocity fluctuations and the time $\tau_t = l/u$ of convective transport of the particles through an inhomogeneity. In particular, the order-of-magnitude estimate of the eddy diffusivity of the magnetic field is

$$\eta_T \sim \overline{u^2}\tau, \quad \tau = \min(\tau_d, \tau_c, \tau_t).$$

It should be stressed that averaging over the distribution P used above differs from usual averaging over an ensemble of realizations of random flow. There may be no turbulence at all and ensemble averaging makes no sense in this case, but averaging over the function P, which in this case is

$$P = \delta(\Delta\boldsymbol{r} - \Delta\boldsymbol{r}_0)\exp\left[-\frac{(\boldsymbol{r}-\boldsymbol{r}_0)^2}{4\eta t}\right]\Big/(4\pi\eta t)^{\frac{3}{2}},$$

is still possible and yields

$$\langle\xi^2\rangle = 6\eta t, \quad \langle\xi\cdot\operatorname{curl}\xi\rangle = \langle\xi^2\operatorname{div}\xi\rangle = 0.$$

Both averagings coincide in the case of infinite conductivities.

This paper considers kinematic aspects of the problem. However, the approach used above can be applied to nonlinear dynamical problems as well if the initial magnetic field has no random fluctuations. (Otherwise, the Lorentz force from fluctuating fields would induce a statistical dependence of the flow on this field and we could not write $\overline{\mathscr{P}\mathscr{A}(\boldsymbol{y},0)} = \overline{\mathscr{P}}\,\overline{\mathscr{A}(\boldsymbol{y},0)}$ to obtain (3.6).) It is natural on physical grounds to expect that the mean-field dynamics reaches a universal regime (i.e. coefficients of the induction equation do not depend explicitly on time at moments of order τ after the initial conditions are specified, independently of the presence or absence of fluctuations of the initial fields). Hence, there seem to be no difficulties of principle in deriving the analogues of (3.13) and (3.17) that will be valid in the nonlinear case. The displacements ξ and the distribution function P will depend implicitly on the magnetic field in this case. Equations (3.13) and (3.17) cannot be applied directly to the nonlinear regime because the Lorentz forces from the mean field inevitably produce the anisotropy in the turbulence (Rüdiger 1974) not allowed for in this paper.

Appendix A. The kinetic equations for fluid particles

In this Appendix we consider the kinetic equations for microscopic distribution functions of fluid particles in the absence of diffusion. In this case the many-particle distribution functions are products of one-particle ones. Hence, it is sufficient to consider the latter.

It is easy to see that the distribution function of the final position x of a particle,

$$q(\boldsymbol{x}\,|\,\boldsymbol{a},t) = \delta(\boldsymbol{x} - \boldsymbol{X}(\boldsymbol{a},t)),$$

satisfies the equation

$$\frac{\partial q}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{u}(\boldsymbol{x},t)\,q = 0 \tag{A 1}$$

with the initial condition

$$q_{t=0} = \delta(\boldsymbol{x}-\boldsymbol{a}). \tag{A 2}$$

Surprisingly, the equation for the distribution function of the initial position $\boldsymbol{y}$ of a particle,

$$f(\boldsymbol{x}\,|\,\boldsymbol{y},t) = \delta(\boldsymbol{y}-\boldsymbol{a}(\boldsymbol{x},t)) \tag{A 3}$$

is not so easy to find. Differentiation of (A 3) with respect to time gives

$$\frac{\partial f}{\partial t} = -\frac{\partial \boldsymbol{a}}{\partial t}\cdot\frac{\partial f}{\partial \boldsymbol{y}}. \tag{A 4}$$

It is obvious that the initial position of a fluid particle does not depend on time; therefore

$$\frac{\mathrm{d}\boldsymbol{a}(\boldsymbol{x},t)}{\mathrm{d}t} = \frac{\partial \boldsymbol{a}(\boldsymbol{x},t)}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{a}(\boldsymbol{x},t) = 0.$$

Consequently

$$-\frac{\partial a_\mu}{\partial t}\frac{\partial f}{\partial y_\mu} = u_\alpha(\boldsymbol{x},t)\frac{\partial a_\mu}{\partial x_\alpha}\frac{\partial f}{\partial y_\mu}. \tag{A 5}$$

It follows from (A 3) that

$$\frac{\partial f}{\partial y_\mu}\frac{\partial a_\mu}{\partial x_\alpha} = -\frac{\partial f}{\partial x_\alpha}. \tag{A 6}$$

Substitution of (A 6) into (A 5) yields

$$\frac{\partial a_\mu}{\partial t}\frac{\partial f}{\partial y_\mu} = u_\mu(\boldsymbol{x},t)\frac{\partial f}{\partial x_\mu}.$$

Finally, we use this relation in the right-hand side of (A 4) to obtain the desired equation:

$$\frac{\partial f}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} f = 0. \tag{A 7}$$

Comparison of (A 1) and (A 7) shows that the distributions of initial and final positions of fluid particles coincide in the case of an incompressible fluid when $\operatorname{div}\boldsymbol{u} = 0$. The two-particle distribution functions also coincide. Coincidence of microscopic distribution functions demands coincidence of the distributions averaged over an ensemble of realizations of a turbulent flow.

Appendix B. Derivation of the equation for the mean field

The distribution function P may be written as

$$P(\boldsymbol{x},\boldsymbol{x}'\,|\,\boldsymbol{y},\boldsymbol{y}',t) = \int \exp\left[\mathrm{i}\boldsymbol{z}\cdot(\boldsymbol{y}-\boldsymbol{x}) + \mathrm{i}\boldsymbol{z}'\cdot(\boldsymbol{y}'-\boldsymbol{x}') + \theta_2(\boldsymbol{z},\boldsymbol{z}',\boldsymbol{\rho},t)\right]\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{z}'/(2\pi)^6, \tag{B 1}$$

where

$$\theta_2 = \ln\langle\exp(\mathrm{i}\boldsymbol{z}\cdot\boldsymbol{\xi}(\boldsymbol{x},t) + \mathrm{i}\boldsymbol{z}'\cdot\boldsymbol{\xi}(\boldsymbol{x}',t))\rangle$$

is the two-particle characteristic function which depends on $\boldsymbol{x}$ and $\boldsymbol{x}'$ and on the difference $\boldsymbol{\rho} = \boldsymbol{x}-\boldsymbol{x}'$ because of the statistical homogeneity of the turbulence considered.

Substitution of (B 1) into (3.6) allows us to differentiate explicitly with respect to $\boldsymbol{x}'$ and pass to the limit $\boldsymbol{x}' \to \boldsymbol{x}$:

$$A_\mu(\boldsymbol{x}, t) = \int \exp\{\mathrm{i}\boldsymbol{z}\cdot\boldsymbol{y} + \mathrm{i}\boldsymbol{z}'\cdot\boldsymbol{y}' - \mathrm{i}\boldsymbol{x}\cdot(\boldsymbol{z}+\boldsymbol{z}') + \theta_2(\boldsymbol{z}, \boldsymbol{z}', 0, t)\}\left(-\mathrm{i}z'_\mu - \left.\frac{\partial\theta_2}{\partial\rho_\mu}\right|_{\rho=0}\right) y'_\alpha A_\alpha(\boldsymbol{y}, 0)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{y}'\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{z}'/(2\pi)^6.$$

It is convenient for what follows to Fourier transform this expression over $\boldsymbol{x}$; subsequent successive integrations over $\boldsymbol{y}$, $\boldsymbol{y}'$, $\boldsymbol{z}'$ and $\boldsymbol{z}$ lead to the relation

$$\hat{A}_\mu(\boldsymbol{k}, t) = \exp[\theta_1(k, t)]\left\{\delta_{\mu\alpha} - \mathrm{i}\left.\frac{\partial^2\theta_2(-\boldsymbol{k}, \boldsymbol{z}, \boldsymbol{\rho}, t)}{\partial z_\alpha\,\partial\rho_\mu}\right|_{\rho=z=0}\right\}\hat{A}_\alpha(\boldsymbol{k}, 0), \tag{B 2}$$

where $\theta_1(\boldsymbol{k}, t) = \theta_2(\boldsymbol{k}, 0, 0, t)$ is the one-particle characteristic function, and

$$\hat{\boldsymbol{A}}(\boldsymbol{k}, t) = \int \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,\boldsymbol{A}(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}/(2\pi)^3$$

is the Fourier image of the vector potential. Let the vector potential to be divergence-free, $\boldsymbol{k}\cdot\hat{\boldsymbol{A}} = 0$, and let the characteristic function to be expanded in a series of semi-invariants (Kliatskin 1980). Equation (B 2) then becomes

$$\hat{A}_\mu(\boldsymbol{k}, t) = \exp[\theta_1(k, t)]\left(a\pi_{\mu\alpha}(\boldsymbol{k}) + \frac{b}{k}\epsilon_{\mu\alpha\nu}k_\nu\right)\hat{A}_\alpha(\boldsymbol{k}, 0), \tag{B 3}$$

where $\pi_{\mu\alpha}(\boldsymbol{k}) = \delta_{\mu\alpha} - k_\mu k_\alpha/k^2$ is a projection tensor,

$$\left.\begin{aligned} a &= 1 - \tfrac{1}{2}\pi_{\mu\alpha}(\boldsymbol{k}) \sum_{p=1}^{\infty} \frac{(-\mathrm{i})^p}{p!}\left\langle \xi_{\sigma_1}\xi_{\sigma_2}\ldots\xi_{\sigma_p}\frac{\partial\xi_\mu}{\partial x_\alpha}\right\rangle_{\mathrm{s}} k_{\sigma_1}k_{\sigma_2}\ldots k_{\sigma_p}, \\ b &= \frac{1}{2k}\epsilon_{\mu\alpha\beta}k_\beta \sum_{p=1}^{\infty} \frac{(-\mathrm{i})^p}{p!}\left\langle \xi_{\sigma_1}\xi_{\sigma_2}\ldots\xi_{\sigma_p}\frac{\partial\xi_\mu}{\partial x_\alpha}\right\rangle_{\mathrm{s}} k_{\sigma_1}k_{\sigma_2}\ldots k_{\sigma_p}. \end{aligned}\right\} \tag{B 4}$$

The sign $\langle\ \rangle_{\mathrm{s}}$ means semi-invariant. For example,

$$\begin{aligned} \left\langle \frac{\partial\xi_\mu}{\partial x_\alpha}\xi_\sigma\xi_\gamma\xi_\beta\right\rangle_{\mathrm{s}} = \left\langle \frac{\partial\xi_\mu}{\partial x_\alpha}\xi_\sigma\xi_\gamma\xi_\beta\right\rangle &- \left\langle \frac{\partial\xi_\mu}{\partial x_\alpha}\xi_\sigma\right\rangle\langle\xi_\gamma\xi_\beta\rangle - \left\langle \frac{\partial\xi_\mu}{\partial x_\alpha}\xi_\gamma\right\rangle\langle\xi_\sigma\xi_\beta\rangle \\ &- \left\langle \frac{\partial\xi_\mu}{\partial x_\alpha}\xi_\beta\right\rangle\langle\xi_\sigma\xi_\gamma\rangle. \end{aligned}$$

Equation (B 3) can be used to express the initial field in terms of the subsequent one:

$$A_\mu(\boldsymbol{k}, 0) = \frac{\exp[-\theta_1(k, t)]}{a^2 + b^2}\left(a\pi_{\mu\alpha}(\boldsymbol{k}) - \frac{b}{k}\epsilon_{\mu\alpha\gamma}k_\gamma\right)\hat{A}_\alpha(\boldsymbol{k}, t). \tag{B 5}$$

Let us differentiate (B 3) with respect to time and exclude the initial field from the resulting equation by use of (B 5). This yields the desired equation for the mean field:

$$\frac{\partial\hat{\boldsymbol{A}}(\boldsymbol{k}, t)}{\partial t} = \left[\dot{\theta}(k, t) + \frac{\dot{a}a + \dot{b}b}{a^2 + b^2}\right]\hat{\boldsymbol{A}}(\boldsymbol{k}, t) + \frac{\dot{a}b - \dot{b}a}{a^2 + b^2}\frac{1}{k}\boldsymbol{k}\times\boldsymbol{A}(\boldsymbol{k}, t), \tag{B 6}$$

where dots above letters signify time derivatives.

Expansion of a, b and θ_1 in powers of the wavenumber k up to terms of third order with allowance for isotropy of the turbulence gives

$$a = 1+\frac{k^2}{12}\langle \xi^2 \operatorname{div} \xi\rangle,$$

$$b = \mathrm{i}\frac{k}{6}\langle \xi\cdot \operatorname{curl} \xi\rangle-\frac{\mathrm{i}k^3}{12}(\tfrac{1}{5}\langle \xi^2\xi\cdot \operatorname{curl}\xi\rangle-\tfrac{1}{3}\langle \xi^2\rangle\, \langle \xi\cdot \operatorname{curl}\xi\rangle),$$

$$\theta_1 = -\frac{k^2}{6}\langle \xi^2\rangle.$$

On collecting in (B 6) the terms of first order in k, we obtain (3.13) for the coefficient α of the alpha-effect. The terms of second and third orders lead to (3.17) and (3.18) respectively. In principle (B 6) offers the possibility of allowing for spatial derivatives of arbitrary order in the equation for the mean field.

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